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Building and Operating the Flexagon

The first flexagon was a hexagonal object folded from a straight strip of paper. This simplest model is quite easily constructed.

Fold off and cut out a sequence of equilateral triangles (see figure 1.1a). Nine of these are to be used in the flexagon. Folding up the flexagon is accomplished by folding consistently clockwise or counterclockwise at every third hinge between triangles (see figure 1.1). If the direction of folding is not consistent, the flexagon will loose both its symmetry and its stability.
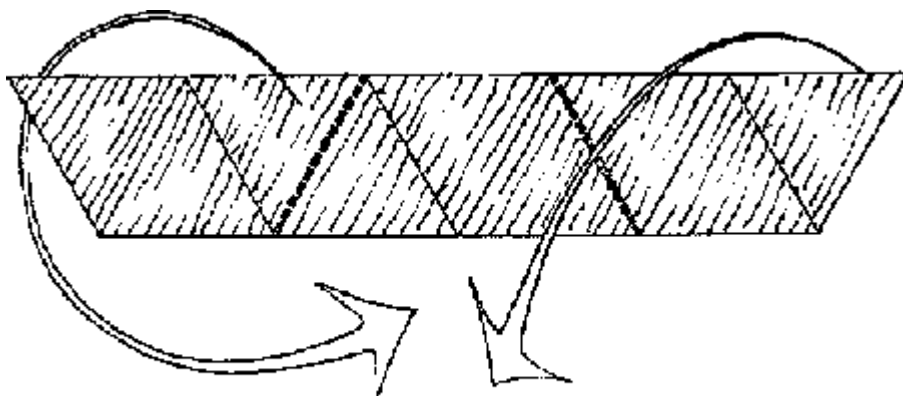

Fig. 1.1a

It will not hold together. When this process
has been completed su-ccessfully and the ends
have been taped in place, the resulting object
should resemble that shown in figure 1.1b.

It can now be seen that this flexagon is a kind
of three twist Moebius band. It has one surface
and one edge. In order to lay it flat, it has
been symmetrically squashed out, giving six
distinct hinged sections, each of which is
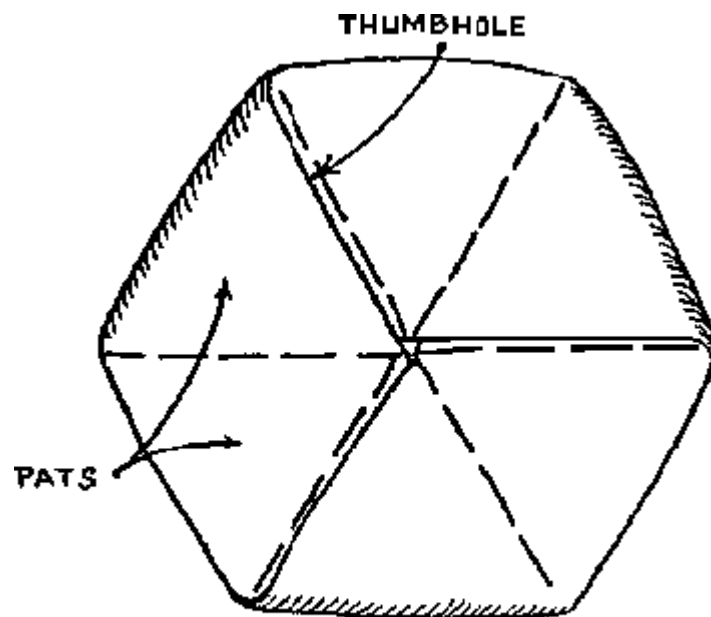called a ``pat".

THUMBHOLE

PATS

Fig. 1.1b

Each triangle of paper making up the pats is
called a ``leaf". It will be found that there
are two kinds of pats. Some of the pats are one
leaf thick, others two. Since the flexagon has
tripole symmetry, any pair of adjacent pats
that one should happen to choose would have one
of each kind of pat. Such a pair will be

referred to as a ``unit".

This is all very interesting, you may say, but what of the supposed ``flexibility"? Before investigating this, we must study the Moebius band a moment longer. Anyone familiar with the Moebius band must know that the twist is in a sense independent of the strip of paper; that is, the twist may be made to pass along the band, or, relative to the twist, the strip may be made to rotate about its open center while the twist remains stationary. The same is the case with our three twist flexagon. We way perform this operation, with some difficulty, after a slight forcing of the ideally rigid leaves. This same operation, carried out in an easier fashion and without bending the leaves, is know as ``flexing".

In order to flex the flexagon, one must first find a hinge between two pats at which the thumb of finger can be passed between the two leaves of the larger pat. Such a hole has been named a ``thumbhole" (see figure 1.1b). When a thumbhole hinge has been found, the two pats on either side should be pushed together. In doing this, fold the pats down away from the thumbhole hinge. It may now be seen that, since alternate hinges are thumbhole hinges, each of the three units may be folded together in this manner. When each units has been folded, the result is a sort of three bladed affair. (see figure 1.2a).
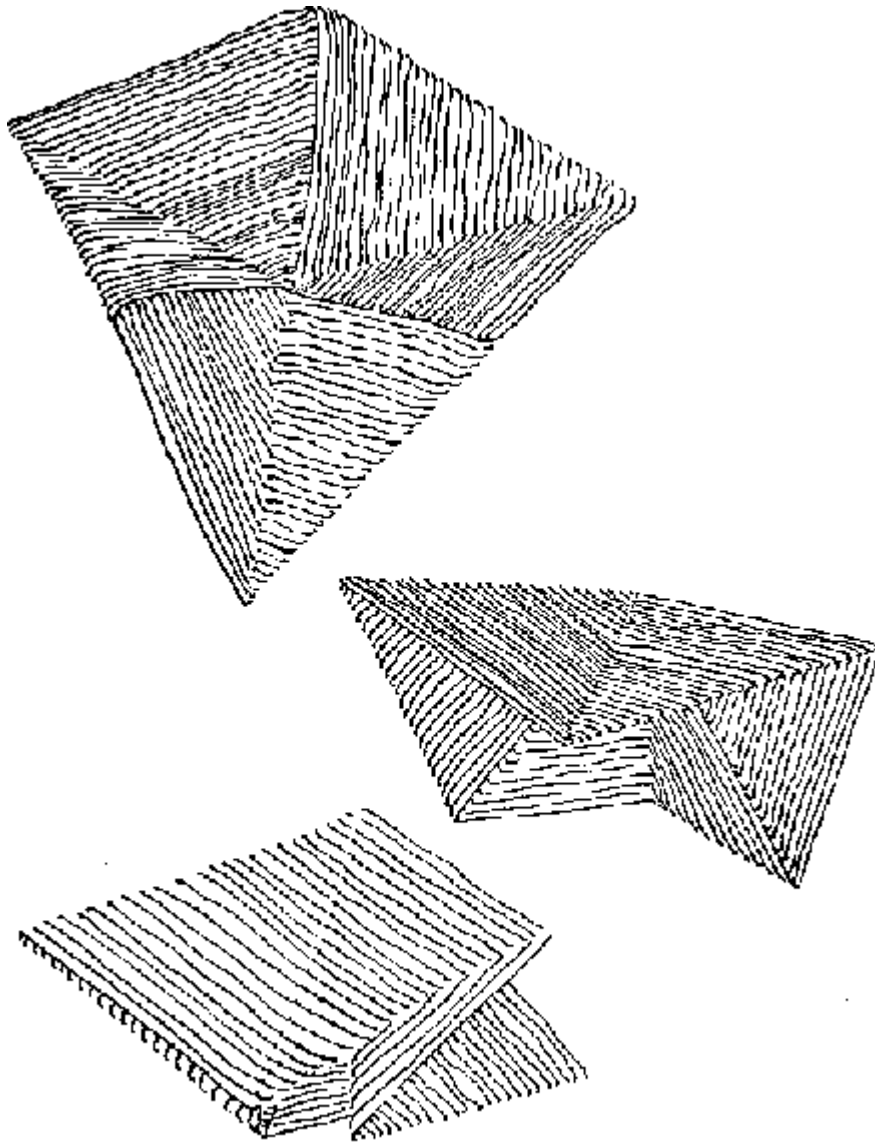
Fig. 1.2a

Now enters the most interesting aspect of the
flexagon. Due to double hinging, our three
bladed object may be opened out again from
either the top or the bottom end. One can, of
course, lay the flexagon flat by performing the
reverse of the process just completed. But this
reverse process may also be carried out from
the downward end of the flexagon, exposing the
faces that had been hidden in the flexagons's
folds (see figure 1.2b). This latter operation
is the essential part of the operation know as

``flexing", and is found to correspond exactly
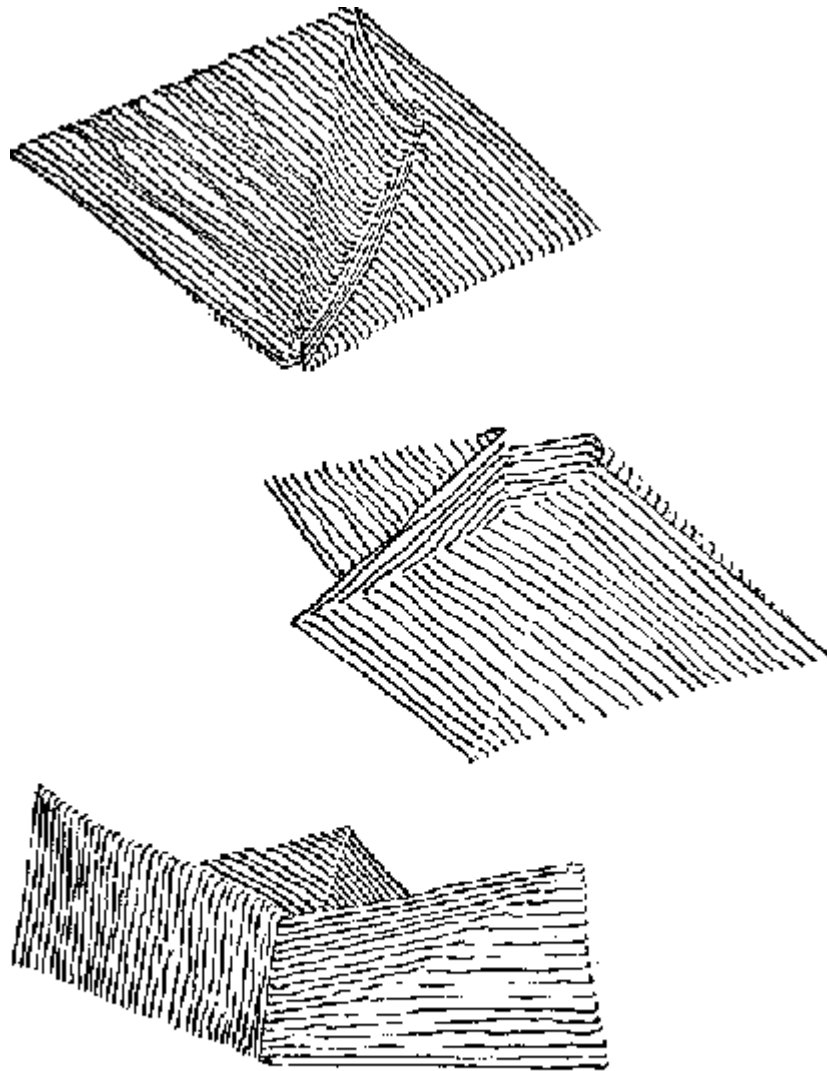to the Moebius band type of operation.

Fig. 1.2b

Now that the flexagon is again flat, is may be
rotated 60° about its center to the right
position for flexing again. This completes the
full ``fle-xing" operation. Thus the process
can be carried out again, and then again. We
can flex in this manner indefinitely, just as
we can perform upon the Moebius band

indefinitely. Facility in flexing may be developed with very little practice.

It is much easier to appreciate the flexagon if we mark it in some way so that we can see just what is happening. A number of interesting marking devices are used. Probably the most common and also the most sensational is color. Several different coloring schemes may be used. We shall use the simplest arrangement possible. Other systems will be shown as the need for them arises. Color the top surfaces of each of the six pats red. Turn the flexagon over and paint the exposed surfaces yellow. Color the remaining surfaces blue after one flexing.

It has not doubt been noticed how carefully it was stressed that the thumbhole hinges be creased <u>upward</u>. The possibility of folding the thum-bhole hinges down, and thereby folding, the other hinges up, has not been considered. Since everything we learn about the flexagon involves breaking former rules, we will of course try doing this. We meet inmediate dis-couragement, however, for we can no longer flex. But since we are not easily dissuaded, we notice that if we were to create a new thumbhole under the thin pats, we might then flex anyway. Indeed the thumbhole seems simply to be a slash sideways through each thick pat. If one could effect such a slitting through the single pats with a sharp knife, it might be possible to create the desired thumbhole. A much more feasible plan involves adding in three extra leaves in the proper places, modeling the new pats on the already-double pats (figure 1.3a). When the color pattern has

been repaired and the long sought after flexing
has been performed, a new surface or, as it is
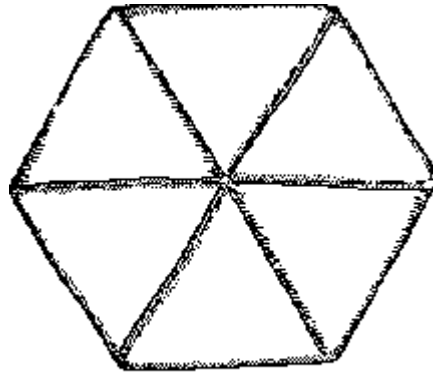called, a new ``side", will be exposed.

Fig. 1.3a

This seems natural enough in itself, but it
leads to an interesting conclusion. Whenever we
come to a position from which we can flex along
only one set of hinges, we can always rotate an
extra $60°$ between flexings, then add in three
new leaves and flex anyway. Therefore, we can
now build a flexagon of any arbitrary size,
with the sides appearing in any desired order.
As a matter of fact, we could say that the
three ``sided" flexagon was derived in this
manner, by slitting, from an ordinary hexagon.

Now that we can build inductively any flexagon,
let us re-examine the flexagon of ``order" four
—that is, the four ``sided" flexagon we
obtained by slitting the first flexagon. The
experimenter may wisely have decided to cut a
hinge and lay the flexagon out flat, exhibiting
what is know as its ``plan". The term ``plan"
refers to the pattern of the paper strip from

wich the flexagon is folded up. In this way can bee seen what has happened to the shape of the strip during the slitting. The strip is no longer straight. It is made up roughly ``C'' – shaped groups of triangles (see figure 1.3b). One triangle of each ``C'' may be folded over an adjacent triangle, similarly in each ``C'', to yield the straight strip of nine triangles that we know can be would up into a three ``sided'' flexagon. This operation is the inverse of the slitting operation.
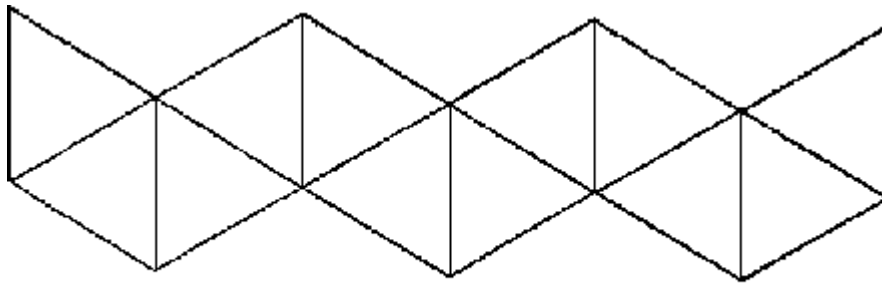

Fig. 1.3b

Now that we have learned the ``plan'' for this particular type of flexagon, we can make as many more like it as we please. If we build new flexagons by ``slitting'', or creating thumbholes, we will be able to increase the number of flexagons whose plans we know how to build to any desired extent.

Some flexagons will be easier to build than others, for the simple reason that their plans will be more easily cut out. There is a certain family of flexagons, for example, all of whose members are formed from straight strips of paper. When we folded the ``C'' of the order

four flexagon into a straight strip, every
third triangle of the straight strip came out
double. If we double the first two in every
three triangles (see figure 1.4), this
corresponds to another slitting and results in
a flexagon of order 5. If all nine triangles
are doubled, the resulting flexagon is of order
6 (figure 1.5a). Opening out this last
flexagon, we see that the uniform doubling of
all the triangles produces another straight
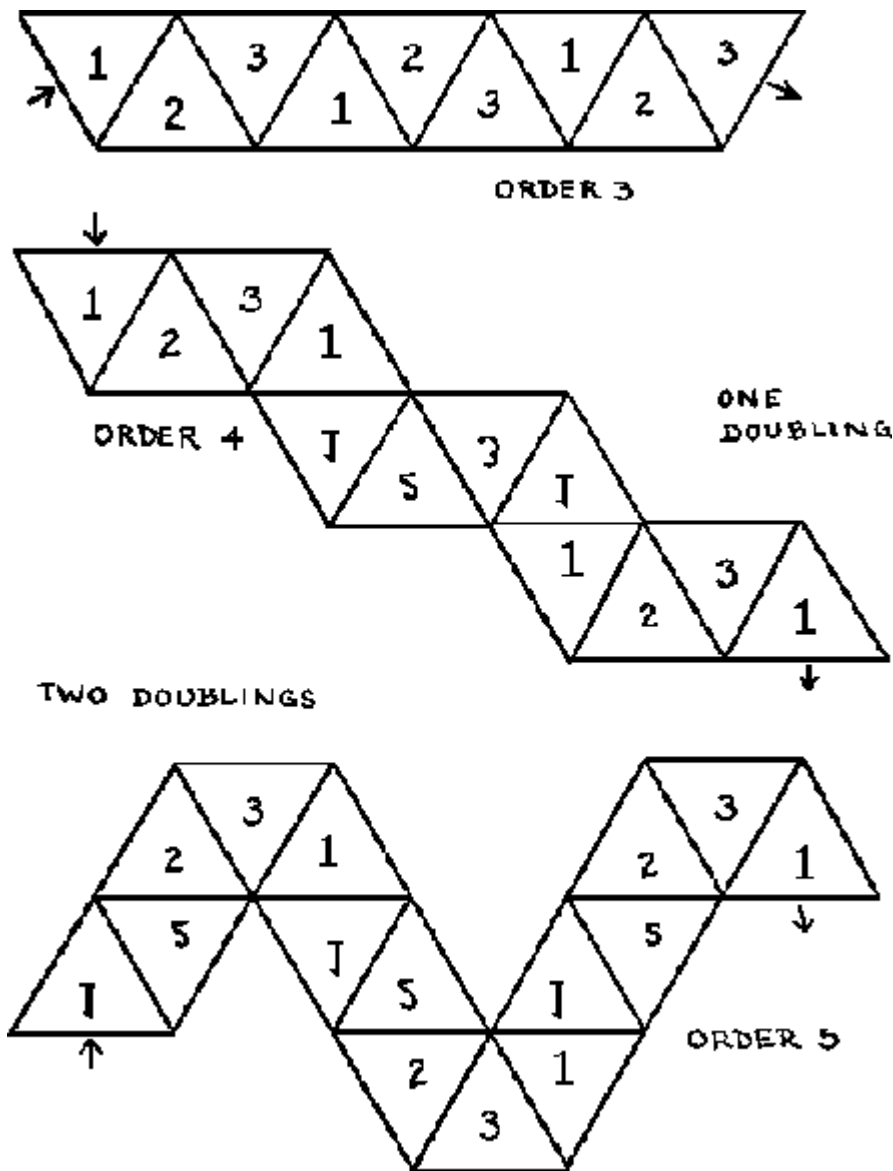strip, twice as long as the first (figure
1.5b).

Fig. 1.4

Are there more ``straight strip'' flexagons?
Clearly we could again double the number of
triangles, obtaining a twelve sided flexagon
made from a straight strip 36 triangles long.
In this case, each of the nine triangles in the
original strip would have become a stack of
four. This doubling process may be continued
until we run out of paper. Each flexagon will
be of order $3 \cdot 2^n$, the number of triangles used
will be $9 \cdot 2^n$, and the number of leaves heaped up

under each of the nine final triangles in the
straight strip will be 2^n. The nth flexagon will
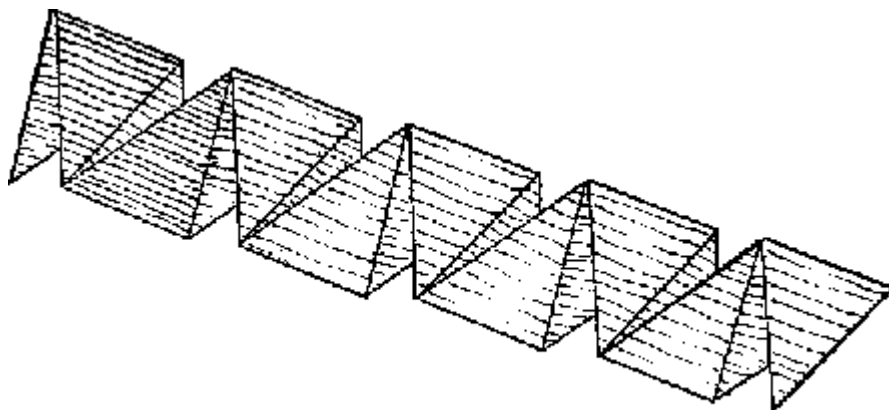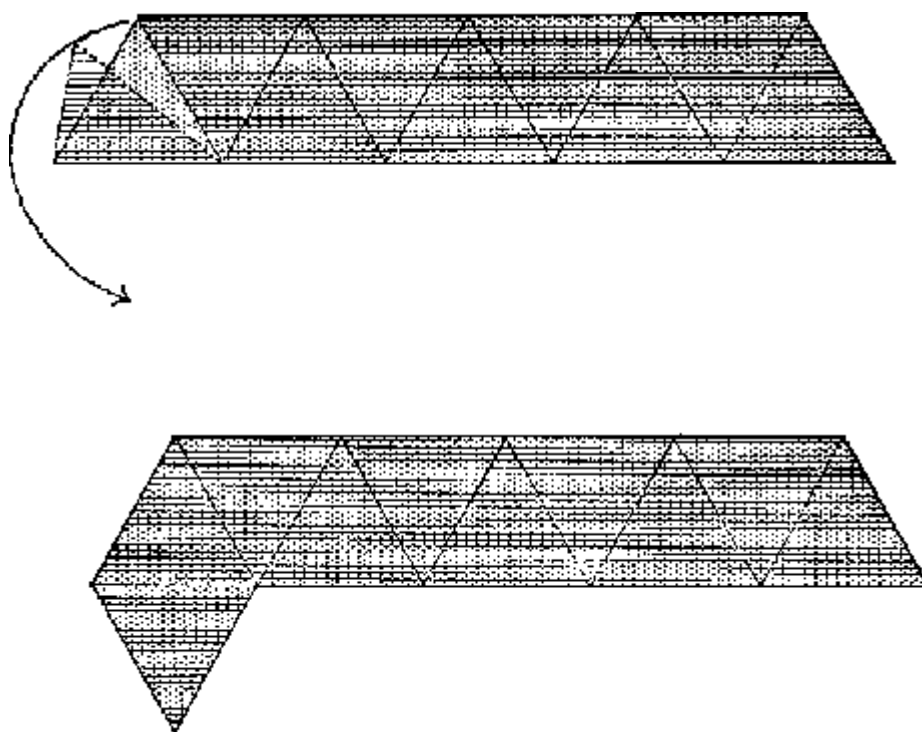always be formed from the $(n - 1)$st.
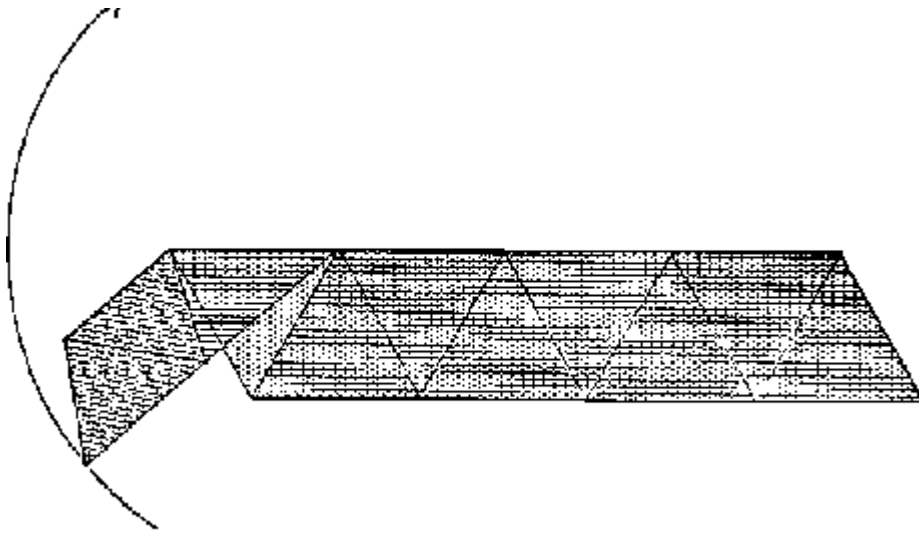

Fig. 1.5a

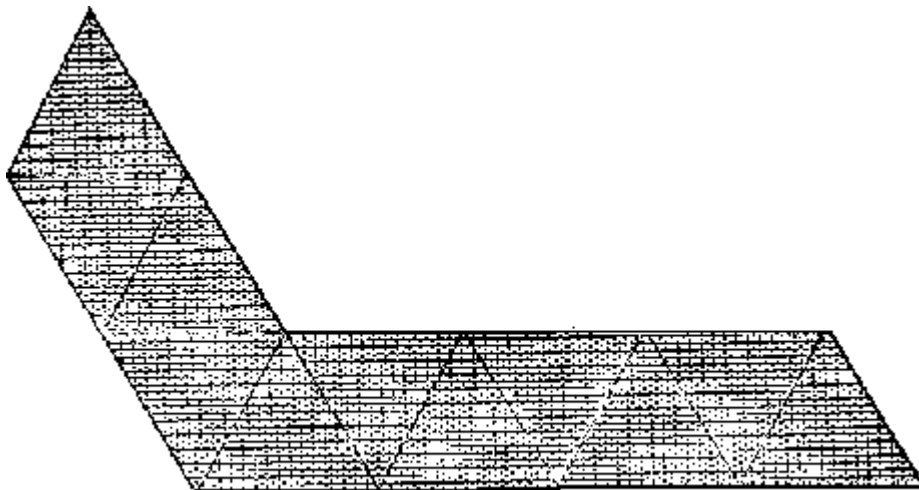Fig. 1.5b

Fig. 1.5b2

Fig. 1.5b3

Let us return to the straight strip of 36
triangles which has been twisted double <u>twice</u>
to form the straight strip of nine heaps (order
= 12). These heaps of four triangles each have
an interesting property that allows us to
create many more straight strip flexagons than
we can using the doubling method alone. We
observe that if we replace a heap of four

triangles by a single leaf, or viceversa, the
straightness of the strip remains unimparied
(see figure 1.6). The significance to be found
in this is that, at any three positions that
will be separated by the same number of leaves
when the ends of the plan have been joined, we
can substitute a heap of four triangles for a
single leaf, and still keep a straight strip
plan. This adds three new sides to the
flexagon. Therefore we can now construct
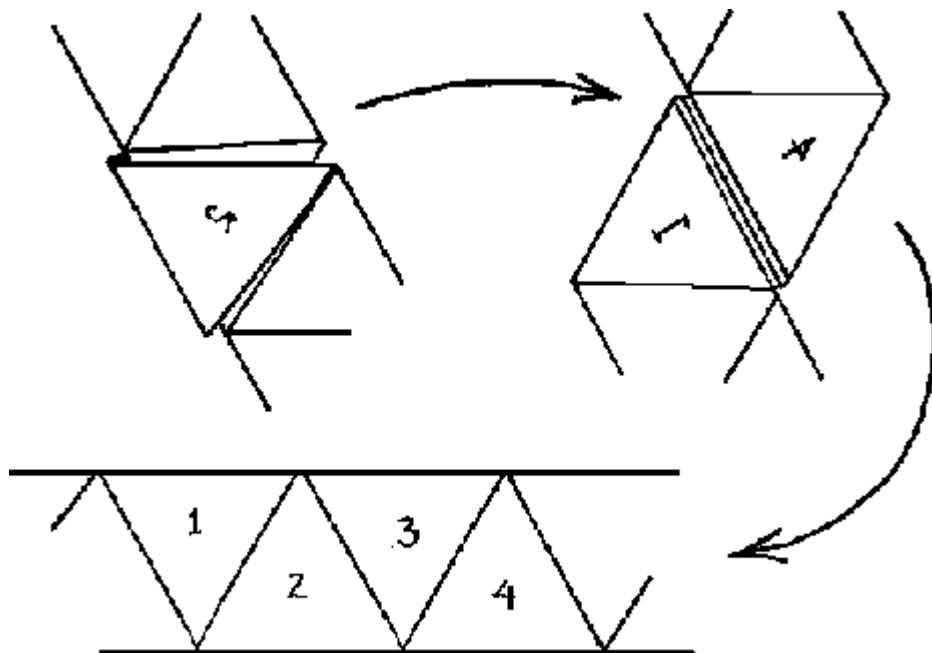flexagons having order of the form 3n from a
straight strip.

Fig. 1.6

An interesting approach to the non-straight
strip flexagon is its consi-deration not as a
slitting product of lower order flexagons, but
as a higher order straight strip flexagon, some
of whose sides have been unslit, i.e., glued
shut.

``Straight strip" flexagons have in general been given the appellation ``regular", although the only things distinguishing them from other, ``irre-gular", flexagons are the ease with which they are built and a certain amount of symmetry which certain of them possess in respect to operation.

Now that any required flexagon can be built, certain purely practical problems begin to arise. Since these may be of interest to the reader, several such details have been brought forth and discussed in Appendix A.

We can at this point perceive a number of elementary relationships that are basic to further study of the flexagon. First, it is easily shown that the total number of leaves must always be three times the order of the flexagon, since every side of every leaf is used. This gives the six leaf faces required for each ``side". Second, every flexagon plan must be reducible by folding to a straight chain of nine piles of triangles, from which the flexagon can be made. To make our third observation, we first place some form of marking device at each of the three corners of one or more of the top leaves of each side, as shown in figure 1.7. As the flexagon is now flexed and turned over, and the various sides are approached via various other sides, the marking devices, or ``dots", indicate that different angles of the leaves are brought to the center of the flexagon at different times. Interesting results have been obtained with this phenomenon through the use of pictures or designs as ``dots"; an example of this due to
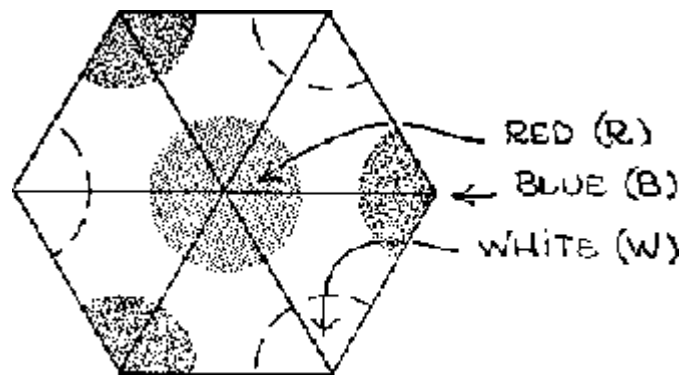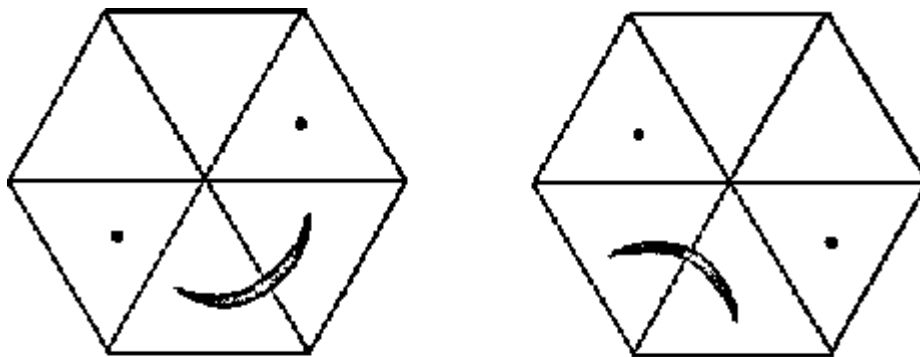
Alan Phillips appears in figure 1.8.

Fig. 1.7

TWO FACES EXHIBITING
THE SAME SIDE

Fig. 1.8

Since there are two different kinds of pats,
arranged alternately about the center of the
flexagon, there are in a sense two different
kinds of leaf faces making up each side. A
flexing will then, fold together pairs of
similarly colored leaf faces, but only one each
of the two types, because it always folds
together pairs of adjacent leaf faces. We are
assuming, of course, that each ``side" is

marked with a single color. Also each color
used in the flexagon is related to at most two
other colors, in the sense that the two will
occupy opposite faces of the same leaf. In
fact, the number will always be two, if the
flexagon has more than two sides. These two
facts lead to the empirically substantiated
deduction that, if one were to fold together
one ``side" of the flexagon, so that only one
color showed, and then read off the colors in
the order in which they would appear as one
passed through each stack of triangles, this
order would be found invariant, for any given
flexagon. The side showing at each position
where this constant order of colors was
recorded would both begin and end the sequence
in each case, but the colors would remain in
the same positions relative to one another.
This important observation should be carefully
noted by the reader, since it provides a
foundation for several later considerations.
The inmediate application is the observation
that, if we are given a properly colored plan,
we can, by folding together adjacent leaf faces
which are colored the same, eventually assemble
the flexagon. Faces of one selected color are
not to be folded together; this will be the
color showing when the process has been
completed.

There remains a relatively well know but much
dreaded aspect of the flexagon. This is the
distortion of the heretofore constant order of
the sides in the flexagon structure, which is
usually accomplished accidentally, and wich
often results in a disassembly and consequent
reassembly of the flexagon involved. The

process is aptly called ``distortion", for it involves forcing of the supposedly rigid leaves. The easiest method of distortion is that in which the flexagon is folded double in a downward direction along a diagonal (see figure 1.9a). The center is the opened out to form a six faced cup (figure 1.9b). The two sets of leaves at one end of the cup are then folded in upon the middle pair (figure 1.9c). The flexagon now opens out disclosing a set of ``sides" in which two pats are of one color and the remaining four of another (see figure 1.9d). This is a specialized operation, and it requires a specialized pat structure. In order to distort a flexagon in this manner, alternate pats must be made up of at least three leaves, with the thumbhole at least two leaves down away from the top of the pat. The remaining pat must be of at least double thickness. The flexagon of order 5 is the simplest flexagon that can be made to meet this requirement. However, a flexagon-like object which will not flex until distorted may be made from a straight strip of ten triangles. After distortion, it becomes of order 3. This is the object shown operating in figure 1.9a, if all the subpats A, B, C, etc. are of a single thickness only. Note that the requirements for doing this operation backwards (which is much harder) are not so demanding. A few more details on the subject of distortions may be found in Appendix D.
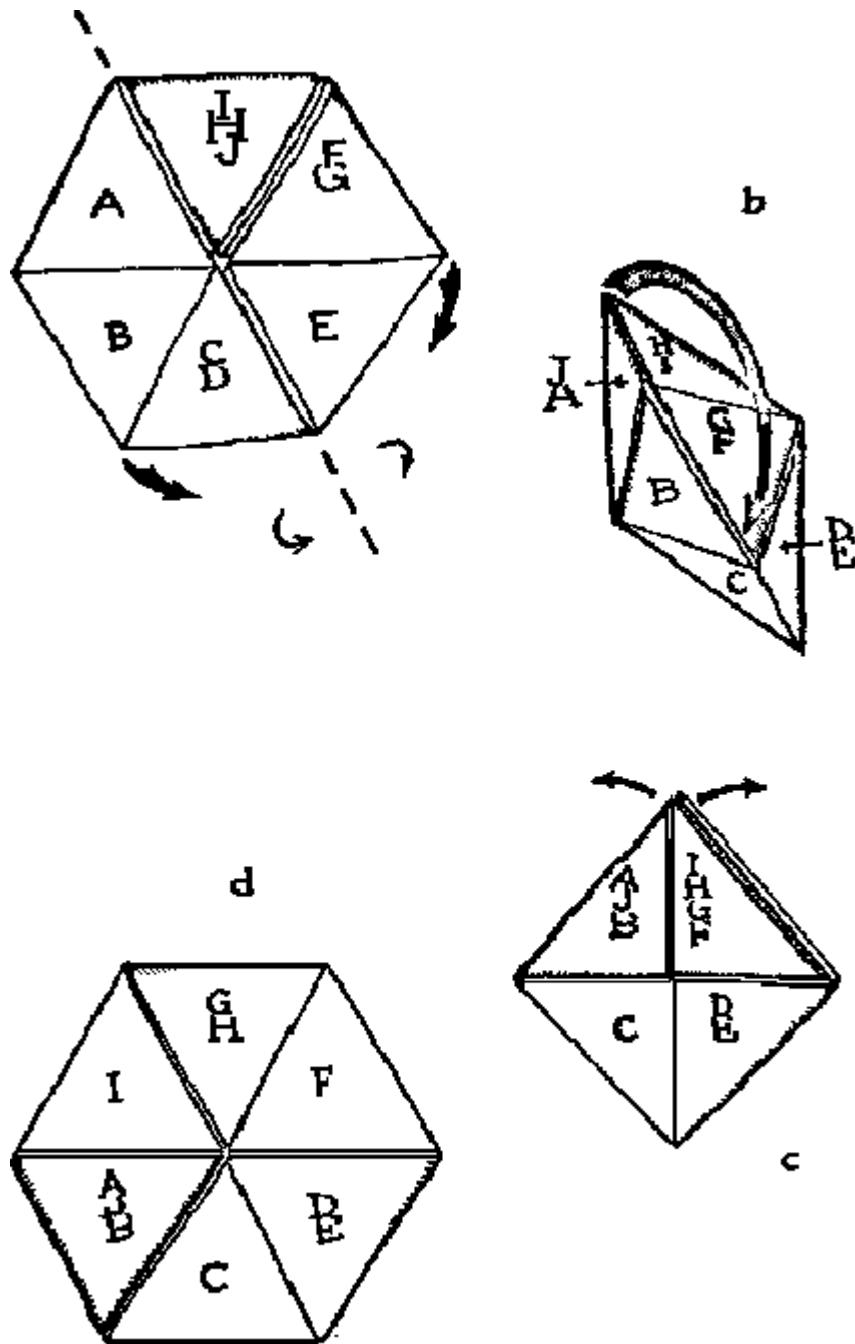
Fig. 1.9

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