FIRST PASSAGE TIME PROBLEM FOR BIASED CONTINUOUS-TIME RANDOM WALKS

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Abstract

We study the first passage time (FPT) problem for biased continuous time random walks. Using the recently formulated framework of fractional Fokker-Planck equations, we obtain the Laplace transform of the FPT density function when the bias is constant. When the bias depends linearly on the position, the full FPT density function is derived in terms of Hermite polynomials and generalized Mittag-Leffler functions.

1. INTRODUCTION

Continuous time random walk (CTRW)[1] is an important model for anomalous diffusion[2] in which the mean square displacement scales with time as $\langle X^2(t) \rangle \sim t^\gamma$ with $0 < \gamma < 2$. In applications of CTRW, one is often interested in its FPT density function.[3] The theory of FPT problems has a long history[4–11] with applications in physics,[12] biology,[13] and engineering.[14]

In an earlier communication,[15] we had described the exact solution for the FPT density function for CTRW (in the diffusion limit) with zero bias. In this paper, we consider the FPT problem for CTRW

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with nonzero bias (both constant and nonconstant). We start with a brief description of the CTRW and write down the fractional Fokker-Planck equation that describes it following Metzler et al.[16] Next we obtain the Laplace transform of FPT density function for a constant bias CTRW. Finally we obtain the full FPT density function when the bias depends linearly on position in terms of Hermite polynomials and generalized Mittag-Leffler functions.[17]

2. CTRW WITH NONZERO BIAS

Consider a one-dimensional CTRW on a discrete space lattice. Denoting the probability of being at site j at time t by $p_j(t)$ and restricting ourselves to nearest neighbor jumps, the CTRW can be described by the following generalized master equation:[16]

$$p_j(t) = \int_0^t dt'[A_{j-1}p_{j-1}(t') + B_{j+1}p_{j+1}(t')]\psi(t - t')$$
$$+ \delta_{x,0}\Phi(t). \qquad (2.1)$$

Here $\psi(t)$ is the waiting time distribution, $\Phi(t) = 1 - \int_0^t dt'\psi(t')$ is the survival probability, A_{j-1} (B_{j+1}) is the probability to jump from site $j - 1$ ($j + 1$) to site j. We consider a biased random walk where the probabilities A_j and B_j are not equal and could depend on the position of the random walker. However, they fulfill the local condition $A_j + B_j = 1$. The waiting time distribution is taken to be a Levy distribution[18] with a power law tail: $\psi(t) \sim (t/\tau)^{-1-\gamma}$ for large t where $0 < \gamma < 1$ and τ has dimensions of time. The biased CTRW described above is subdiffusive i.e. its mean squared displacement varies with time t as t^γ where $0 < \gamma < 1$.

Following Metzler et al.,[16] we can now transition to continuous space and introduce the Taylor series expansion:

$$A_{j-1}p_{j-1}(t) \sim A(x)p(x, t) - \Delta x \frac{\partial A(x)p(x, t)}{\partial x}$$
$$+ \frac{(\Delta x)^2}{2}\frac{\partial^2 A(x)p(x, t)}{\partial x^2} \qquad (2.2)$$

with a similar expansion for $B_{j+1}p_{j+1}(t)$. We substitute the above expansions in the generalized master equation and Laplace transform it in time. Further, we take the generalized diffusion limit:

$\Delta x, \tau \to 0$ such that

$$F(x) = \lim_{\Delta x, \tau \to 0} \frac{\Delta x}{\tau^\gamma}[A(x) - B(x)] \qquad (2.3)$$

$$K = \lim_{\Delta x, \tau \to 0} \frac{(\Delta x)^2}{2\tau^\gamma}. \qquad (2.4)$$

Here $F(x)$ is the force acting on the particle during every jump and K is the generalized diffusion constant. Due to the presence of the force $F(x)$, the biased CTRW is also known as "CTRW with (non)constant force".[16] The Laplace transformed generalized master equation can be inverted (see Metzler et al.[16] for details) to finally give the following fractional Fokker-Planck equation (FFPE) for the probability density function $W(x, t)$ of the biased CTRW in the generalized diffusion limit:

$$W(x, t) - W(x, 0)$$
$$= {}_0D_t^{-\gamma}\left[K\frac{\partial^2}{\partial x^2}W(x, t) - \frac{\partial}{\partial x}F(x)W(x, t)\right],$$
$$0 < \gamma < 1. \qquad (2.5)$$

Here we have incorporated the initial condition $W(x, 0)$ and the Riemann-Liouville fractional integral operator ${}_0D_t^{-\gamma}$ acting on a function $g(t)$ is defined as[19,20]

$${}_0D_t^{-\gamma}g(t) = \frac{1}{\Gamma(\gamma)}\int_0^t dt'(t - t')^{\gamma-1}g(t'), \quad \gamma > 0. \qquad (2.6)$$

Here $\Gamma(z)$ is the usual gamma function.[21] We call the diffusive process obtained from the CTRW as "Levy type of anomalous diffusion" due to the waiting time distribution characteristics.[15]

We now formulate the FPT problem for the FFPE given in Eq. (2.5). This problem can be recast as a boundary value problem with absorbing boundaries.[3] In our case, to obtain the FPT density function, we first need to solve Eq. (2.5) with the following boundary and initial conditions:

$$W(0, t) = 0, \quad W(\infty, t) = 0,$$
$$W(x, 0) = \delta(x - a) \qquad (2.7)$$

where $x = a$ is the starting point of the CTRW, containing the initial concentration of the distribution. Once we solve for $W(x, t)$, the FPT density $f(t)$ is given by[3]

$$f(t) = -\frac{d}{dt}\int_0^\infty dx\, W(x, t). \qquad (2.8)$$

3. FIRST PASSAGE TIME FOR CONSTANT BIAS

In this section, we study the FPT problem for Levy type anomalous subdiffusion with constant bias $-\nu$ (that is, $F(x) = -\nu$). The Laplace transform of the FPT density is obtained.

For constant bias, the FFPE to be solved is given by [cf. Eq. (2.5)]

$$W(x, t) - W(x, 0)$$
$$= {}_0D_t^{-\gamma}\left[\nu\frac{\partial}{\partial x}W(x, t) + K\frac{\partial^2}{\partial x^2}W(x, t)\right],$$
$$0 < \gamma < 1 \qquad (3.1)$$

with the boundary and initial conditions given in Eq. (2.7). We solve the FFPE using the method of separation of variables.[22] Let $W(x, t) = X(x)T(t)$. Substituting in Eq. (3.1) we obtain

$$X(x)T(t) - X(x)$$
$$= [{}_0D_t^{-\gamma}T(t)][\nu X'(x) + KX''(x)] \qquad (3.2)$$

where the primes denote the derivatives with respect to x. Separating out the variables and introducing the separation constant λ we get

$$KX''(x) + \nu X'(x) = -\lambda X(x) \qquad (3.3)$$

and

$$T(t) - 1 = -\lambda \, {}_0D_t^{-\gamma}T(t). \qquad (3.4)$$

First we solve Eq. (3.4). Taking its Laplace transform, we obtain

$$T(s) - \frac{1}{s} = \frac{\lambda}{s^\gamma}T(s). \qquad (3.5)$$

Here we have used the fact that the Laplace transform of ${}_0D_t^{-\gamma}T(t)$ is $T(s)/s^\gamma$. Solving for $T(s)$, we get

$$T(s) = \frac{1}{s - \lambda s^{1-\gamma}}. \qquad (3.6)$$

Taking the inverse Laplace transform,[17] we finally obtain

$$T(t) = E_\gamma[-\lambda t^\gamma] \qquad (3.7)$$

where $E_\gamma(z)$ is the Mittag-Leffler function[17] with the following power series expansion:

$$E_\gamma(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(1 + \gamma n)}. \qquad (3.8)$$

Note that $E_\gamma(z)$ reduces to the regular exponential function when $\gamma = 1$.

Next consider Eq. (3.3). The solution of this equation satisfying the boundary conditions is given by

$$X(x) = \exp[-\nu(x-a)/2K]\frac{\sin[x\sqrt{\lambda/K - \nu^2/4K^2}]}{2\sqrt{K\lambda - \nu^2/4}},$$
$$\lambda \geq \nu^2/4K. \qquad (3.9)$$

Thus we have a continuous spectrum for λ. Combining the solutions for $X(x)$ and $T(t)$, $W(x, t)$ is given by the following integral over λ:

$$W(x, t) = \frac{2}{\pi}\int_{\nu^2/4K}^\infty d\lambda A(\lambda)\exp[-\nu(x-a)/2K]\frac{\sin[x\sqrt{\lambda/K - \nu^2/4K^2}]}{2\sqrt{K\lambda - \nu^2/4}}E_\gamma[-\lambda t^\gamma]. \qquad (3.10)$$

The coefficient $A(\lambda)$ is fixed by the initial condition ($W(x, 0) = \delta(x - a)$) and we get

$$W(x, t) = \frac{2}{\pi}\int_{\nu^2/4K}^\infty d\lambda \exp[-\nu(x-a)/2K]\sin[a\sqrt{\lambda/K - \nu^2/4K^2}]\frac{\sin[x\sqrt{\lambda/K - \nu^2/4K^2}]}{2\sqrt{K\lambda - \nu^2/4}}E_\gamma[-\lambda t^\gamma]. \qquad (3.11)$$

Letting $\lambda' = \sqrt{\lambda/K - \nu^2/4K^2}$, we obtain

$$W(x, t) = \frac{2}{\pi}\int_0^\infty d\lambda' \sin\lambda' a \sin\lambda' x \exp[-\nu(x-a)/2K]E_\gamma[-(K\lambda'^2 + \nu^2/4K)t^\gamma]. \qquad (3.12)$$

Using standard trigonometric identities and dropping the primes, the above equation can be rewritten as

$$W(x, t) = \frac{1}{\pi}\int_0^\infty d\lambda \cos\lambda(x-a)\exp[-\nu(x-a)/2K]E_\gamma[-(K\lambda^2 + \nu^2/4K)t^\gamma]$$

$$- \frac{1}{\pi}\int_0^\infty d\lambda \cos\lambda(x+a)\exp[-\nu(x-a)/2K]E_\gamma[-(K\lambda^2 + \nu^2/4K)t^\gamma]. \qquad (3.13)$$

Taking the Laplace transform with respect to time, we get

$$q(x,\, s) = \frac{1}{\pi} \int_0^\infty d\lambda \cos \lambda(x - a) \exp[-\nu(x - a)/2K] \frac{1}{s + \nu^2 s^{1-\gamma}/4K + K\lambda^2 s^{1-\gamma}}$$

$$- \frac{1}{\pi} \int_0^\infty d\lambda \cos \lambda(x + a) \exp[-\nu(x - a)/2K] \frac{1}{s + \nu^2 s^{1-\gamma}/4K + K\lambda^2 s^{1-\gamma}} \qquad (3.14)$$

where $q(x,\, s)$ is the Laplace transform of $W(x,\, t)$. Here we have used the fact that the Laplace transform of $E_\gamma(-Bt^\gamma)$ is $1/(s + Bs^{1-\gamma})$.[17]

We can now perform the integration over λ by using the following result:[23]

$$\int_0^\infty \frac{\cos \lambda x}{\alpha^2 + \lambda^2} = \frac{\pi}{2\alpha} e^{-\alpha|x|}. \qquad (3.15)$$

Thus we obtain

$$q(x,\, s) = \frac{1}{2\sqrt{K}} \exp[-\nu(x - a)/2K] \frac{s^{\gamma-1}}{\sqrt{s^\gamma + \nu^2/4K}}$$

$$\times \{\exp[-\sqrt{s^\gamma + \nu^2/4K} \, |x - a|/\sqrt{K}]$$

$$- \exp[-\sqrt{s^\gamma + \nu^2/4K}(x + a)/\sqrt{K}]\}. \qquad (3.16)$$

To obtain the Laplace transform $F(s)$ of the FPT density function $f(t)$, we take the Laplace transform of Eq. (2.8) to get

$$F(s) = -s \int_0^\infty dx\, q(x,\, s) + \int_0^\infty dx\, W(x,\, 0). \qquad (3.17)$$

Here we have used the fact that Laplace transform of $dW(x,\, t)/dt$ is given by $sq(x,\, s) - W(x,\, 0)$.[23] Since $W(x,\, 0) = \delta(x - a)$ [cf. Eq. (2.7)], we obtain

$$F(s) = 1 - s \int_0^\infty dx\, q(x,\, s). \qquad (3.18)$$

Substituting for $q(x,\, s)$ from Eq. (3.16), we get

$$F(s) = 1 - \frac{1}{2\sqrt{K}} \frac{s^\gamma}{\sqrt{s^\gamma + \nu^2/4K}} \int_0^\infty dx$$

$$\times \exp[-\nu(x - a)/2K]$$

$$\times \{\exp[-\sqrt{s^\gamma + \nu^2/4K} \, |x - a|/\sqrt{K}]$$

$$- \exp[-\sqrt{s^\gamma + \nu^2/4K} \, (x + a)/\sqrt{K}]\}. \qquad (3.19)$$

After considerable manipulation, we obtain the final result upon evaluating the integrals to be

$$F(s) = \exp[a(\nu - \sqrt{\nu^2 + 4Ks^\gamma})/2K],$$
$$0 < \gamma \leq 1. \qquad (3.20)$$

We now prove that $F(s)$ is the Laplace transform of a valid probability distribution by showing that $F(0) = 1$ and $F(s)$ is completely monotone:[24]

$$(-1)^n \frac{d^n F(s)}{ds^n} \geq 0, \quad \forall n,\, s > 0. \qquad (3.21)$$

It is easily verified that $F(0) = 1$. We next prove that $F(s)$ is completely monotone for $0 < \gamma \leq 1$. $F(s)$ can be written as the product $F_1(s)F_2(s)$ where $F_1(s) = \exp(a\nu/2K)$ and $F_2(s) = \exp(-\rho(s))$ with

$$\rho(s) = \sqrt{\nu^2 + 4Ks^\gamma}/2K. \qquad (3.22)$$

From Feller,[24] $F_2(s)$ is completely monotone if $\rho(s)$ is a positive function (for $s > 0$) with a completely monotone derivative. It is obvious that $\rho(s)$ is a positive function. The derivative $\rho'(s)$ is given by

$$\rho'(s) = \frac{\gamma s^{\gamma-1}}{\sqrt{\nu^2 + 4Ks^\gamma}}. \qquad (3.23)$$

Now $\rho'(s)$ can be written as the product $\rho'_1(s)\rho'_2(s)$ where

$$\rho'_1(s) = \gamma s^{\gamma-1}, \quad \rho'_2(s) = 1/\sqrt{\nu^2 + 4Ks^\gamma}. \qquad (3.24)$$

The first factor $\rho'_1(s)$ is completely monotone since

$$(-1)^n \frac{d\rho'_1(s)}{ds^n}$$

$$= (-1)^n \gamma(\gamma - 1) \cdots (\gamma - n)s^{\gamma-n-1} \geq 0 \qquad (3.25)$$

for $0 < \gamma < 1$. Similarly, $\rho'_2(s)$ is also completely monotone. Now the product of two completely monotone functions is also completely monotone.[24] Hence $\rho'(s)$ is completely monotone. This implies that $F_2(s)$ is completely monotone. Trivially, $F_1(s)$ is also completely monotone. Therefore the product $F(s) = F_1(s)F_2(s)$ is completely monotone for

$0 < \gamma < 1$. Consequently, $F(s)$ is the Laplace transform of a valid probability distribution.

From the above expressions for $F(s)$ and its derivatives, one can calculate the moments of the FPT distribution.[24] We find that all the moments diverge for $0 < \gamma < 1$.

The above solution for $F(s)$ can be extended to include $\gamma = 1$ (ordinary Brownian motion). Specifically, for $\gamma = 1$, the inverse Laplace transform of $F(s)$ can be carried out to give

$$f(t) = \frac{a}{\sqrt{4\pi K t^3}} \exp\left[-\frac{(a - \nu t)^2}{4Kt}\right], \quad (3.26)$$

$$a > 0, \quad t > 0.$$

This is nothing but the FPT density function for a Brownian motion with drift ν.[25] This can be seen as follows. For $\gamma = 1$, the FFPE in Eq. (3.1) reduces to:

$$W(x, t) - W(x, 0)$$

$$= {}_0D_t^{-1}\left[\nu\frac{\partial}{\partial x}W(x, t) + K\frac{\partial^2}{\partial x^2}W(x, t)\right]. \quad (3.27)$$

Differentiating both sides with respect to t we get

$$\frac{\partial}{\partial t}W(x, t) = \nu\frac{\partial}{\partial x}W(x, t) + K\frac{\partial^2}{\partial x^2}W(x, t). \quad (3.28)$$

This is the usual Fokker-Planck equation for ordinary Brownian motion with drift ν thus explaining the observed FPT density for $\gamma = 1$. In other words, for a ordinary Brownian motion, a constant bias is equivalent to a constant drift.

4. FIRST PASSAGE TIME FOR LINEAR BIAS

In this section, we study the first passage time problem for Levy type anomalous subdiffusion with bias $-\omega x$ which depends linearly on the position x (that is $F(x) = -\omega x$). An expression for the FPT density is obtained in terms of Hermite polynomials and generalized Mittag-Leffler functions.[17]

In this case, the FFPE to be solved is given by [cf. Eq. (2.5)]

$$W(x, t) - W(x, 0)$$

$$= {}_0D_t^{-\gamma}\left[\frac{\partial}{\partial x}(\omega x W(x, t)) + K\frac{\partial^2}{\partial x^2}W(x, t)\right],$$

$$0 < \gamma < 1. \quad (4.1)$$

with

$$W(0, t) = W(\infty, t) = 0,$$

$$W(x, 0) = \delta(x - a). \quad (4.2)$$

We solve this FFPE again using the method of separation of variables.[22] Let $W(x, t) = X(x)T(t)$. Substituting in Eq. (4.1) we obtain

$$X(x)T(t) - X(x)$$

$$= [{}_0D_t^{-\gamma}T(t)][\omega x X'(x) + \omega X(x) + K X''(x)], \quad (4.3)$$

where the primes denote derivatives with respect to x. Separating out the variables and introducing the separation constant λ we get

$$KX''(x) + \omega x X'(x) + \omega X(x) = -\lambda X(x) \quad (4.4)$$

and

$$T(t) - 1 = -\lambda \, {}_0D_t^{-\gamma}T(t). \quad (4.5)$$

The above equations were solved for natural boundary conditions ($W(-\infty, t) = W(\infty, t) = 0$) by Metzler et al.[16] We follow a similar approach here. Let

$$Y(\tilde{x}) = e^{-\tilde{x}^2/2}X(\tilde{x}), \quad \tilde{x} = \sqrt{\omega/K}\, x. \quad (4.6)$$

Substituting in Eq. (4.4) and assuming $\omega \neq 0$ we get

$$Y''(\tilde{x}) - \tilde{x}Y'(\tilde{x}) + \frac{\lambda}{\omega}Y(\tilde{x}) = 0. \quad (4.7)$$

This is nothing but the differential equation satisfied by Hermite polynomials $H_n(\tilde{x}/\sqrt{2})$. In our case we have the boundary conditions $Y(0) = Y(\infty) = 0$. But it can be easily shown that any function satisfying these boundary conditions can be represented in terms of the odd degree Hermite polynomials $H_{2n+1}(\tilde{x}/\sqrt{2})$. Further, these polynomials satisfy the following orthogonality condition which follows in a straightforward fashion from the usual orthogonality condition[21] satisfied over the domain $(-\infty, \infty)$:

$$\int_0^\infty dx\, H_{2n+1}(x)H_{2m+1}(x)e^{-x^2}$$

$$= \sqrt{\pi}2^{2n}(2n + 1)!\delta_{nm}. \quad (4.8)$$

Thus the equation for $Y(\tilde{x})$ becomes an eigenvalue equation with the eigenvalue

$$\lambda_{2n+1} = (2n+1)\omega, \quad n = 0, 1, \ldots, \qquad (4.9)$$

and the eigenfunction

$$Y_{2n+1}(\tilde{x}/\sqrt{2})$$

$$= \frac{1}{2^n \pi^{1/4} \sqrt{(2n+1)!}} H_{2n+1}(\tilde{x}/\sqrt{2}). \quad (4.10)$$

In terms of the original variables we have

$$X_{2n+1}(x)$$

$$= \frac{1}{2^n \pi^{1/4} \sqrt{(2n+1)!}} e^{-\omega x^2/2K} H_{2n+1}(\sqrt{\omega/2K}x).$$
$$(4.11)$$

We have already solved the equation for $T(t)$ in the previous section. The only difference here is that λ takes on discrete values given by Eq. (4.9):

$$T_{2n+1}(t) = E_\gamma[-(2n+1)\omega t^\gamma] \qquad (4.12)$$

where $E_\gamma(z)$ is the Mittag-Leffler function defined earlier. The general solution for $W(x, t)$ can now be written down:

$$W(x, t) = \sum_{n=0}^{\infty} A_{2n+1} \frac{1}{2^n \pi^{1/4} \sqrt{(2n+1)!}} e^{-\omega x^2/2K}$$

$$\times H_{2n+1}(\sqrt{\omega/2K}x) E_\gamma[-(2n+1)\omega t^\gamma].$$
$$(4.13)$$

The coefficients A_{2n+1} are determined by imposing the initial condition $W(x, 0) = \delta(x-a)$. Using the orthogonality condition for $H_{2n+1}(x)$ given in Eq. (4.8), we can easily determine A_{2n+1} to be

$$A_{2n+1}$$

$$= \sqrt{\omega/2K} \frac{1}{2^n \pi^{1/4} \sqrt{(2n+1)!}} H_{2n+1}(\sqrt{\omega/2K}a).$$
$$(4.14)$$

Substituting this in the expression for $W(x, t)$ we get

$$W(x, t) = \sqrt{\omega/2\pi K} \sum_{n=0}^{\infty} \frac{1}{4^n (2n+1)!}$$

$$\times H_{2n+1}(\sqrt{\omega/2K}a) e^{-\omega x^2/2K}$$

$$\times H_{2n+1}(\sqrt{\omega/2K}x) E_\gamma[-(2n+1)\omega t^\gamma].$$
$$(4.15)$$

To obtain the FPT distribution, we substitute the above expression for $W(x, t)$ in Eq. (2.8). Using the following result[21]

$$\int_0^\infty dx e^{-\omega x^2/2K} H_{2n+1}(\sqrt{\omega/2K}x)$$

$$= \sqrt{2K/\omega}(-1)^n \frac{4^n}{\sqrt{\pi}} \Gamma(n+1/2) \qquad (4.16)$$

we get

$$f(t) = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+1/2)}{\Gamma(2n+2)}$$

$$\times H_{2n+1}(\sqrt{\omega/2K}a) \frac{d}{dt} E_\gamma[-(2n+1)\omega t^\gamma].$$
$$(4.17)$$

From the series expansion of Mittag-Leffler function Eq. (3.8), we have

$$\frac{d}{dt} E_\gamma[-(2n+1)\omega t^\gamma]$$

$$= \sum_{k=0}^{\infty} \frac{\gamma k[-(2n+1)\omega t^\gamma]^{k-1}[-(2n+1)\omega t^{\gamma-1}]}{\Gamma(1+\gamma k)}.$$
$$(4.18)$$

Using the facts that the summand vanishes for $k = 0$ and $\Gamma(1+x) = x\Gamma(x)$, we have

$$\frac{d}{dt} E_\gamma[-(2n+1)\omega t^\gamma]$$

$$= \sum_{k=1}^{\infty} \frac{\gamma k[-(2n+1)\omega t^\gamma]^{k-1}[-(2n+1)\omega t^{\gamma-1}]}{\gamma k \Gamma(\gamma k)}.$$
$$(4.19)$$

Letting $k' = k - 1$ and simplifying, we get

$$\frac{d}{dt} E_\gamma[-(2n+1)\omega t^\gamma]$$

$$= -(2n+1)\omega t^{\gamma-1} \sum_{k'=0}^{\infty} \frac{[-(2n+1)\omega t^\gamma]^{k'}}{\Gamma(\gamma k' + \gamma)}.$$
$$(4.20)$$

But the infinite sum is the series expansion of the generalized Mittag-Leffler function $E_{\gamma,\gamma}[-(2n+1)\omega t^\gamma]$.[17] Therefore we have

$$\frac{d}{dt} E_\gamma[-(2n+1)\omega t^\gamma]$$

$$= -(2n+1)\omega t^{\gamma-1} E_{\gamma,\gamma}[-(2n+1)\omega t^\gamma]. \quad (4.21)$$

Substituting this in the expression for $f(t)$, we finally get

$$f(t) = \frac{\omega}{\sqrt{\pi}t^{1-\gamma}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} H_{2n+1}(\sqrt{\omega/2K}a)$$

$$\times E_{\gamma,\gamma}[-(2n+1)\omega t^{\gamma}]. \tag{4.22}$$

Here we have also used the following doubling rule[21] for gamma functions to simplify the expression for $f(t)$:

$$\Gamma(2n+2) = (2n+1)\Gamma(2n+1)$$

$$= (2n+1)4^n\Gamma(n+1/2)\Gamma(n+1)/\sqrt{\pi}. \tag{4.23}$$

5. CONCLUSIONS

In this paper we studied the FPT problem for a biased CTRW. Solution to this problem was obtained using the recently formulated FFPE which describes the CTRW in the (generalized) diffusion limit. Expressions for the FPT density function was obtained in two cases — constant bias and bias that depends linearly on the position.

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